Examples of the Atiyah-Singer Index Theorem

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These notes are about two examples of the Atiyah-Singer index theorem. In the first one we look at the n-sphere (for an even n) with its unique spin structure and the second one treats the Clifford bundle $\bigwedge^* T^*S^2 \otimes \mathbb{C}$ over S^2 equipped with the Euler grading. For the second example we need an extension of the originally Atiyah-Singer index theorem for general graded Clifford bundles. First the classical statement from Atiyah and Singer [see Roe99, p.164]:

Theorem 1 (Atiyah-Singer)**.** *Let M be a compact, even-dimensional oriented manifold and let S be a canonically graded Clifford bundle over it with associated Dirac operator D. Then*

$$Ind(D) = \int_M \hat{\mathcal{A}}(TM) \wedge ch(S/\Delta) \tag{1}$$

holds. In particular, if M is a spin manifold and $S = \Delta$ is the spin bundle, the index of the Dirac operator on Δ is equal to the $\hat{\mathcal{A}}$-genus of the manifold M.

A Clifford bundle S equipped with a general grading (with grading operator ϵ) splits into a direct sum of the canonical and anticanonically graded Clifford subbundles $S = S_c \oplus S_a$. Let ϵ_0 be the grading operator of the canonical grading on S, then ϵ_0 and $-\epsilon_0$ are the grading operators on S_c and S_a. Explicitly this means:

$$\epsilon(s_c) = \epsilon_0(s_c) \quad \text{and} \quad \epsilon(s_a) = -\epsilon_0(s_a) \quad \forall s_c \in S_c,\ s_a \in S_a \tag{2}$$

The fact, that we can always find such a splitting [Lemma 11.3 Roe99], gives us the extended Atiyah-Singer index theorem for a general grading on S:

Corollary 2 (Atiyah-Singer index theorem for a general grading)**.** *The index of the associated Dirac operator D of a graded Clifford bundle S over a compact, even-dimensional oriented manifold M satisfies:*

$$Ind(D) = \int_M \hat{\mathcal{A}}(TM) \wedge ch_s(S/\Delta)$$

Here the relative super Chern character is defined as $ch_s(S/\Delta) =: ch(S_c/\Delta) - ch(S_a/\Delta)$ with the splitting $S = S_c \oplus S_a$ into the canonical and anticanonically graded parts of S.

Proof. Lets take any notions of the corollary and the paragraph before. Splitting all vector bundles into the even and odd parts gives for the Clifford bundle S

$$S = S_c^+ \oplus S_c^- \oplus S_a^+ \oplus S_a^- = S^+ \oplus S^-.$$

A straightforward calculation gives $S_c^+ \oplus S_a^- = S^+$ and $S_c^- \oplus S_a^+ = S^-$. For example is $S_c^+ \oplus S_a^- \subset S^+$:

> Pick an arbitrary $s_c + s_a \in S_c^+ \oplus S_a^-$ and calculate under use of equation (2) $\epsilon(s_c + s_a) = \epsilon(s_c) + \epsilon(s_a) = \epsilon_0(s_c) - \epsilon_0(s_a) = s_c + s_a$. It follows $s_c + s_a \in S^+$.

Now we use the analog notation for the restricted Dirac operators and the corollary follows with the definition of the index and the originally Atiyah-Singer index theorem:

$$\begin{aligned}
\mathrm{Ind}(D) &= \dim(\ker(D^+)) - \dim(\ker(D^-)) \\
&= \dim(\ker(D_c^+)) + \dim(\ker(D_a^-)) - \dim(\ker(D_c^-)) - \dim(\ker(D_a^+)) \\
&= \mathrm{Ind}(D_c) - \mathrm{Ind}(D_a) \\
&= \int_M \hat{\mathcal{A}}(TM) \wedge \mathrm{ch}(S_c/\Delta) - \int_M \hat{\mathcal{A}}(TM) \wedge \mathrm{ch}(S_a/\Delta) \\
&= \int_M \hat{\mathcal{A}}(TM) \wedge \mathrm{ch}_s(S/\Delta)
\end{aligned}$$

□

First example: n-Sphere with spin structure

As manifold we choose for an even integer n the n-sphere denoted as S^n. This is a compact, even-dimensional oriented manifold which carries a unique spin structure*. The induced spin bundle Δ forms a Clifford bundle [see Roe99, p.63] and the requirements of the Atiyah-Singer index theorem are fulfilled. It states for the associated Dirac operator D:

$$\boxed{\mathrm{Ind}(D) = \int_{S^n} \hat{\mathcal{A}}(TS^n)} \tag{3}$$

In the following we want to calculate both sides separately and verify the index theorem. For the calculation of the index of the Dirac operator the following lemma will be helpful:

Lemma 3. *The associated Dirac operator D of a compact spin manifold M with positive scalar curvature has no homogeneous spinors. This means explicitly that the equation $D\phi = 0$ has just the trivial solution.*

Proof. The spin manifold M induces the spin bundle Δ which carries the structure of a Clifford bundle. D is the associated Dirac operator to this Clifford bundle. The twisting curvature of the spin bundle Δ is zero by Proposition 4.21 and together with Proposition 3.18 [see Roe99, p.64 and 48] the square of the Dirac operator takes the form

$$D^2 = \nabla^*\nabla + \frac{1}{4}\kappa \tag{4}$$

with scalar curvature κ. Let ϕ be a smooth section of Δ which satisfies $D\phi = 0$. Integrating over the manifold M (here we need the compactness of M), using that D is self-adjoin and that ∇^* is the formal adjoin of ∇ gives us

$$0 = \int_M \langle D\phi, D\phi\rangle d\mathrm{vol}_g = \int_M \langle D^2\phi, \phi\rangle d\mathrm{vol}_g \overset{\text{eq.(4)}}{=} \int_M \underbrace{\langle \nabla^*\nabla\phi, \phi\rangle}_{=\|\nabla\phi\|^2} d\mathrm{vol}_g + \frac{1}{4}\int_M \kappa\|\phi\|^2 d\mathrm{vol}_g$$

*For $n > 2$ S^n is spin and it is 2-connected because all homotopy groups $\pi_k(S^n)$ vanishes for $k < n$ and with Proposition 4.17 [see Roe99, p.63] the existence of a unique spin-structure follows. In the case $n = 2$ there is also a unique spin structure [see DT86].

and it follows $\phi \equiv 0$ because of the positive scalar curvature. $\square$

Let's go back to equation (3) and calculate both sides:

Left side of eq. (3): The spin bundle Δ is canonically graded and splits into the positive and negative half-spin representations $\Delta_+ \oplus \Delta_-$ with Dirac operator $D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}$. The n-sphere has positive scalar curvature so that we can apply the previous Lemma 3, which gives us $\ker(D^\pm) = 0$. It follows by the definition of the index:

$$\mathrm{Ind}(D) = \dim(\ker(D^+)) - \dim(\ker(D^-)) = 0$$

Right side of eq. (3): The $\hat{\mathcal{A}}$-genus of the real vector bundle TS^n is defined over the Pontrjagin class

$$\hat{\mathcal{A}}(TS^n) = \exp\left(\bigwedge_{\log(f)} (\log(p(TS^n))\right)$$

with $f(x) = \frac{\sqrt{x}/2}{\sinh(\sqrt{x}/2)}$ [see Bär06, p.20]. To see that the Pontrjagin genus of the n-sphere is equal to one we take the normal bundle ϵ of S^n, which is a trivial line bundle and satisfies $TS^n \oplus \epsilon \cong R^{n+1}|_{S^n}$. From the standard properties of the Pontrjagin genus [Satz 3.2 Bär06] we can conclude:

$$p(TS^n) = p(TS^n) \cdot p(\epsilon) = p(TR^{n+1}) = 1 \in H^{4*}(M, R) \tag{5}$$

It follows $\hat{\mathcal{A}}(TS^n) = 1$ and finally $\int_{S^n} \hat{\mathcal{A}}(TS^n) = 0$ because the integral over a differential form with lower degree than n is zero.

Both sides of equation (3) are zero and the Atiyah-Singer index theorem is verified for our example.

Second example: $\bigwedge^* \mathbf{T}^*\mathbf{S}^2 \otimes \mathbb{C}$ over $\mathbf{S}^2$ equipped with the Euler grading

Let $\bigwedge^* T^*S^2 \otimes \mathbb{C}$ be the Clifford bundle over S^2 [Example 3.19 Roe99] equipped with the Euler grading [Example 11.8 Roe99]. We want to consider the generalization of the Atiyah-Singer index theorem (Corollary 2). The following theorem is an application of this statement. It identifies the Euler characteristic $\chi(M)$ with the Euler class and is known as the Chern-Gauss-Bonnet theorem [Question 13.20 Roe99]:

Theorem 4. *(Chern-Gauss-Bonnet theorem) The Clifford bundle $S = \bigwedge^* T^*M \otimes \mathbb{C}$ over a compact, even-dimensional oriented manifold M, equipped with the Euler grading, satisfies:*

$$\chi(M) := \sum_j (-1)^j dim(H^j(M, \mathbb{R})) = \int_M e(TM)$$

Remark 5. (1) The vector bundle $S = \bigwedge^* T^*M \otimes \mathbb{C}$ inherits the structure of a Clifford bundle by using the natural isomorphism to $Cl(TM) \otimes \mathbb{C}$ ($e_1 \wedge ... \wedge e_n \mapsto e_1 \cdot ... \cdot e_n$ for $e_1, ..., e_n$ orthonormal and identifying T^*M with TM via the metric). Then we define the module action such that the following diagram commutes:

$$\begin{array}{ccc} (Cl(TM)\otimes\mathbb{C})\times\left(\bigwedge^* T^*M\otimes\mathbb{C}\right) & \xrightarrow{\text{module action}} & \bigwedge^* T^*M\otimes\mathbb{C} \\ \sim\downarrow & & \downarrow\sim \\ (Cl(TM)\otimes\mathbb{C})\times(Cl(TM)\otimes\mathbb{C}) & \xrightarrow{\text{Clifford multipl.}} & Cl(TM)\otimes\mathbb{C} \end{array}$$

In Example 3.19 [see Roe99, p.49] it is proved, that this gives us really the structure of a Clifford bundle with Dirac operator $D = d + d^*$.

(2) In the previous theorem we take the definition of the Euler characteristic used in algebraic topology. It corresponds to the geometric definition for a manifold via a triangulation. If M is a surface (two dimensional manifold) we have:

$$\chi(M) := \sum_j (-1)^j \dim(H^j(M,\mathbb{R})) = V - E + F \tag{6}$$

Here is V the number of vertexes, E the number of edges and F the number of surfaces of a triangulation.

Proof. (Chern-Gauss-Bonnet theorem) The idea of the proof is to start with the extended version of the Atiyah-Singer index theorem (Corollary 2) and consider both sides of the statement with the expressions in the Chern-Gauss-Bonnet theorem:

(1) $\operatorname{Ind}(D) = \sum_j (-1)^j \dim(H^j(M,\mathbb{R}))$

(2) $\int_M \hat{\mathcal{A}}(TM) \wedge \operatorname{ch}_s(S/\Delta) = \int_M e(TM)$

We will start with the index of the Dirac operator. The grading operator ϵ of the Euler grading is defined for an element of the form $w \otimes z \in \wedge^j T^*M \otimes \mathbb{C}$ via $\epsilon(w \otimes z) := (-1)^j w \otimes z$. This leads to the following splitting of our Clifford bundle:

$$S = \bigoplus_j \underbrace{\bigwedge\nolimits^j T^*M \otimes \mathbb{C}}_{:=S_j} = \underbrace{(\bigwedge\nolimits^{\text{even}} T^*M \otimes \mathbb{C})}_{:=S_+} \bigoplus \underbrace{\left(\bigwedge\nolimits^{\text{odd}} T^*M \otimes \mathbb{C}\right)}_{:=S_-}$$

Using the map $d \otimes id$ between the $\Gamma(S_j)$ this leads to a Dirac complex in the sense of Definition 6.1 [see Roe99, p.87]. Under use of the Hodge theorem [Theorem 6.2 Roe99] we can calculate the kernel of the Dirac operator restricted to $\Gamma(S_j)$:

$$\ker\left(D|_{\Gamma(S_j)}\right) = \{s \in \Gamma(S_j) | \underbrace{\text{s is harmonic}}_{\Leftrightarrow Ds=0}\} \overset{\text{Hodges theorem}}{\cong} H^j(S; d \otimes id) \cong H^j(M;\mathbb{R}) \tag{7}$$

After this preliminary work, we can calculate the index of the Dirac operator:

$$\begin{aligned} \operatorname{Ind}(D) &= \dim(\ker(D_+)) - \dim(\ker(D_-)) \\ &= \sum_j \dim\Big(\underbrace{\ker\left(D|_{\Gamma(S_{2j})}\right)}_{\cong H^{2j}(M;\mathbb{R})}\Big) - \sum_j \dim\Big(\underbrace{\ker\left(D|_{\Gamma(S_{2j+1})}\right)}_{\cong H^{2j+1}(M;\mathbb{R})}\Big) \\ &\overset{\text{eq.(7)}}{=} \sum_j (-1)^j \dim(H^j(M;\mathbb{R})) \end{aligned} \tag{8}$$

For the explicit calculation of the integral over the $\hat{\mathcal{A}}$-genus and the super Chern character we must know precisely how the canonical and anticanonically graded parts of S looks like.

Claim: $S_c \cong \Delta \otimes \Delta_+$ and $S_a \cong \Delta \otimes \Delta_-$ (9)

Proof: Recall that we have the two grading operators $\epsilon = (-1)^j$ (acting like this on elements of $\bigwedge^j T^*M \otimes \mathbb{C}$) and $\epsilon_0 = i^{n/2}\omega$ (where n is the dimension of the manifold M and $\omega = e_1 \cdot \ldots \cdot e_n$ is the volume element in $Cl(TM)$) on S, the natural isomorphism $S \cong Cl(TM) \otimes \mathbb{C}$ and the isomorphism (the spin representation) $\kappa : Cl(TM) \otimes \mathbb{C} \to \mathrm{End}(\Delta)$ [see Roe99, p.61]. The plan is to find $\tilde{\epsilon}$ and $\tilde{\epsilon_0}$ such that the following diagram commutes:

$$\begin{array}{ccccc} S & \overset{\sim}{=\!=\!=} & Cl(TM) \otimes \mathbb{C} & \xrightarrow{\kappa} & \mathrm{End}(\Delta) \\ {\scriptstyle \epsilon \backslash \epsilon_0} \downarrow & & & & \downarrow {\scriptstyle \tilde{\epsilon} \backslash \tilde{\epsilon_0}} \\ S & \overset{\sim}{=\!=\!=} & Cl(TM) \otimes \mathbb{C} & \xrightarrow{\kappa} & \mathrm{End}(\Delta) \end{array} \tag{10}$$

Then we can do the splitting into the canonical and anticanonically graded parts of S for $\mathrm{End}(\Delta)$ instead for S: Define $\tilde{\epsilon}(A) := f \circ A \circ f$ and $\tilde{\epsilon_0}(A) := f \circ A$ for $A \in \mathrm{End}(\Delta)$ where $f := i^{n/2}\kappa(e_1 \cdot \ldots \cdot e_n \otimes 1) \in \mathrm{End}(\Delta)$ is the involution which eigenspaces define the positive and negative half-spin representations $\Delta_\pm$ [see Roe99, p.62] [see FNS00, p.22]. The diagram (10) commutes for this $\tilde{\epsilon}$ and $\tilde{\epsilon_0}$ because for an homogeneous element $s \in \bigwedge^j T^*M \otimes \mathbb{C}$ we have:

$$\tilde{\epsilon}(\kappa(s)) = f \circ \kappa(s) \circ f = (-1)^{n/2}\kappa(e_1 \cdot \ldots \cdot e_n \cdot \underbrace{s \cdot e_1 \cdot \ldots \cdot e_n}_{=(-1)^j e_1 \cdot \ldots \cdot e_n \cdot s}) = \kappa((-1)^j s) = \kappa(\epsilon(s))$$
$$\tilde{\epsilon_0}(\kappa(s)) = f \circ \kappa(s) = \kappa(i^{n/2} e_1 \cdot \ldots \cdot e_n \cdot s) = \kappa(\epsilon_0(s))$$

Based on the definition of $\Delta_\pm$ as eigenspaces of f, the splitting into the canonical and anticanonically graded parts $\mathrm{End}(\Delta) = \mathrm{Hom}(\Delta_+, \Delta) \oplus \mathrm{Hom}(\Delta_-, \Delta)$ follows (here we interpret $\mathrm{Hom}(\Delta_\pm, \Delta)$ as a subspace of $\mathrm{End}(\Delta)$ under use of the trivial extension $\Delta_\mp \mapsto 0 \in \Delta$):

- $\forall A \in \mathrm{Hom}(\Delta_+, \Delta)$: $\tilde{\epsilon}(A) = f \circ A \circ f = f \circ A = \epsilon_0(A)$
- $\forall A \in \mathrm{Hom}(\Delta_-, \Delta)$: $\tilde{\epsilon}(A) = f \circ A \circ f = -(f \circ A) = -\epsilon_0(A)$

Now the Claim is shown because of the following natural identifications:

- $S_c \cong \mathrm{Hom}(\Delta_+, \Delta) \cong \Delta_+^* \otimes \Delta \cong \Delta \otimes \Delta_+$
- $S_a \cong \mathrm{Hom}(\Delta_-, \Delta) \cong \Delta_-^* \otimes \Delta \cong \Delta \otimes \Delta_-$

With the previous result we can write out the super Chern character explicitly:

$$\mathrm{ch}_s(S/\Delta) \underset{(9)}{\overset{\text{Def.}}{=}} \mathrm{ch}\big((\Delta \otimes \Delta_+)/\Delta\big) - \mathrm{ch}\big((\Delta \otimes \Delta_-)/\Delta\big) = \mathrm{ch}_s(\Delta) = e(TM) \tag{11}$$

The last step holds because of the calculations in exercise 4.34 [see Roe99, p.69]. The Chern-

Gauss-Bonnet theorem follows:

$$\sum_j (-1)^j \dim(H^j(M;\mathbb{R})) \overset{(8)}{=} \operatorname{Ind}(D) \overset{\text{Corollary 2}}{=} \int_M \hat{\mathcal{A}}(TM) \wedge \operatorname{ch}_s(S/\Delta) \overset{(11)}{=} \int_M e(TM) \qquad \square$$

Together with the previous theorem the equation in Corollary 2 takes, for our special case $M = S^2$, the form:

$$\boxed{\chi(S^2) := \sum_i (-1)^i \dim(H^i(S^2, \mathbb{R})) = \int_{S^2} e(TS^2)} \tag{12}$$

We will calculate both sides separately:

Left side of eq. (12): Using the geometric definition of the Euler characteristic, the triangulation shown in Figure 1 gives us:

$$\chi(S^2) = V - E + F = 6 - 12 + 8 = 2$$

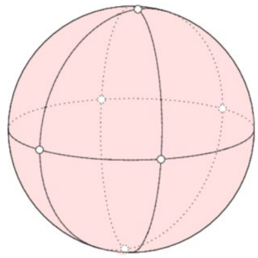

Figure 1: Triangulation of the 2-sphere [Goo].

Right side of eq. (12): The Euler class of the real oriented vector bundle TS^2 is equal to the first Chern class of the line bundle $T_{\mathbb{C}}S^2$ [Question 2.36(v) Roe99]. Here we identify any real two dimensional fiber of TS^2 with a one dimensional complex vector space (the orientation must be preserved) and glue it together to a complex line bundle denoted as $T_{\mathbb{C}}S^2$. The plan is to calculate the curvature matrix Ω with values in the 2-forms for a connection on $T_{\mathbb{C}}S^2$ (Ω is a 1x1 matrix because $T_{\mathbb{C}}S^2$ is a line bundle) and obtain the first Chern class via $c_1(T_{\mathbb{C}}S^2) = \left[\frac{-1}{2\pi i}\operatorname{tr}(\Omega)\right]$ [Definition 2.21 Roe99]. We use spherical coordinates (θ, φ) on S^2 and choose the Levi-Civita connection ∇ on TS^2. Then the metric and the connection takes the following form:

$$\begin{aligned} &g = d\theta \otimes d\theta + \sin^2(\theta) d\varphi \otimes d\varphi; \\ &\nabla_{\partial_\theta}\partial_\theta = 0; \quad \nabla_{\partial_\varphi}\partial_\theta = \frac{\cos(\theta)}{\sin(\theta)}\partial_\varphi; \quad \nabla_{\partial_\varphi}\partial_\varphi = \sin(\theta)\cos(\theta)\partial_\theta \end{aligned} \tag{13}$$

For any point $p \in S^2$ we identify $\partial_\theta|_p = 1$ and $\frac{1}{\sin(\theta)}\partial_\varphi|_p = i$ such that we can interpret ∂_θ as a smooth section of $T_{\mathbb{C}}S^2$ which is at every point in the domain of the spherical coordinates linear independent. A short calculation gives for two vector fields $X = X^\varphi \partial_\varphi + X^\theta \partial_\theta, Y = Y^\varphi \partial_\varphi + Y^\theta \partial_\theta \in \Gamma(TS^2)$ under use of the relations in equation (13) and $\partial_\varphi = i\sin(\theta)\partial_\theta$

$$R(X,Y)\partial_\theta = \left(X^\varphi Y^\theta - X^\theta Y^\varphi\right)\partial_\varphi = i\left(X^\varphi Y^\theta \sin(\theta) - X^\theta Y^\varphi \sin(\theta)\right)\partial_\theta \tag{14}$$

such that the curvature matrix with values in the 2-forms looks like $\Omega = i\sin(\theta) d\varphi \wedge d\theta$. Now we can puzzle everything together and calculate the integral over the Euler class:

$$\int_{S^2} e(TS^2) = \int_{S^2} c_1(T_{\mathbb{C}}S^2) = \frac{-1}{2\pi i}\int_{S^2} \operatorname{tr}(\Omega) = \frac{1}{2\pi i}\int_{S^2} i\sin(\theta) d\theta \wedge d\varphi = \frac{1}{2\pi}4\pi = 2$$

We have just calculated the Euler class on the domain of the spherical coordinates picked in the beginning. But there is only one point missing, who doesn't play a role in the calculation of the integral.

Both sides of equation (12) gives the same result and the Chern-Gauss-Bonnet theorem (which is an application of the Atiyah-Singer index theorem) is verified for this example.

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